A characterization of $L(2,1)$-labeling number for trees with maximum degree 3*

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Abstract

An $L(2,1)$-labeling of a graph is an assignment of nonnegative integers to the vertices of G such that adjacent vertices receive numbers differed by at least 2, and vertices at distance 2 are assigned distinct numbers. The $L(2,1)$-labeling number is the minimum range of labels over all such labeling. It was shown by Griggs and Yeh [Labelling graphs with a condition at distance 2, *SIAM J. Discrete Math.* 5(1992), 586-595] that the $L(2,1)$-labeling number of a tree is either $\Delta+1$ or $\Delta+2$. In this paper, we give a complete characterization of $L(2,1)$-labeling number for trees with maximum degree 3.

Keywords: $L(2,1)$-labelling, characterization, tree, distance two.

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1 Introduction

An $L(2,1)$-labeling f of a graph G is a function from the vertex set $V(G)$ to the set of nonnegative integers such that $|f(x)-f(y)| \geq 2$ if x and y are adjacent and $|f(x)-f(y)| \geq 1$ if x and y are at distance 2, where $x, y \in V(G)$. A k-$L(2,1)$-labeling of a graph is an $L(2,1)$-labeling with image at most k. The $L(2,1)$-labeling number of G, denoted by $\lambda(G)$, is the smallest k such that G has a k-$L(2,1)$-labeling.

The $L(2,1)$-labeling of a graph arose from a variation of the Frequency Channel Assignment problem introduced by Hale [7]. This subject has been studied rather extensively [1–6, 8–10, 12]. It is obvious that $\lambda(G) \geq \Delta+1$ for any graph G, where Δ is the maximum degree of G. For the upper bound of $\lambda(G)$, Griggs and Yeh [6] proved $\lambda(G) \leq \Delta^2+2\Delta$. Moreover, they conjectured that $\lambda(G) \leq \Delta^2$ for any graph G with $\Delta \geq 2$ and they confirmed the conjecture for a few classes of graphs such as paths, cycles, trees, graphs with diameter 2, etc. In 1996, Chang and Kuo [1] proved that $\lambda(G) \leq \Delta^2+\Delta$ for any graph G. Král and Škrekovski [9] improved this bound by showing that $\lambda(G) \leq \Delta^2+\Delta-1$. In 2005, Daniel Gonçalves [5] proved that $\lambda(G) \leq \Delta^2+\Delta-2$ and this remains the best upper bound until now.

Let T be a tree, Griggs and Yeh [6] proved that $\lambda(T)$ is either $\Delta+1$ or $\Delta+2$. It is easy to see that there exist infinitely many trees T such that $\lambda(T)=\Delta+1$ or $\lambda(T)=\Delta+2$. Wang [11] and

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Zhai [13] established some sufficient conditions for $\lambda(T) = \Delta + 1$. Naturally, it is very interesting to give a characterization for trees to have different $L(2,1)$-labeling numbers. In this paper, we give a solution for the case of $\Delta = 3$.

2 Structural analysis

For a tree T, a vertex of degree k is called a k-vertex. Let $V_k(T)$ be the set of k-vertices in T. A vertex is called *major* if $d(v) = \Delta$; *minor* if $d(v) < \Delta$; a *leaf* if $d(v) = 1$, and a *handle* if $d(v) > 1$ and v is adjacent to at least $d(v) - 1$ leaves. A *k-handle* is a handle of degree k. A path $P = ux_1x_2 \cdots x_kv$ is called a *k-uv-chain* (*k-chain* or *uv-chain*) if $d(u) \neq 2$, $d(v) \neq 2$ and $d(x_i) = 2$ for all $i = 1, 2, \ldots, k$. In particular, $P = uv$ is defined as a 0-chain. We denote a directed k-uv-chain by $[u(k)v]$ (or $[uv]$ for short) if it is a directed path starting from u to v, and both u and v are major vertices. Similarly, a directed k-uv-chain $(u(k)v]$ (or $(uv]$ for short) if u is a leaf and v is a major vertex. Moreover, $(uv]$ is called *open* and $[uv]$ is called *closed*. Note that closed uv-chain has two orientations $[uv]$ and $[vu]$.

A major vertex is called *generalized major handle* if it is incident to at least $\Delta - 1$ open chains. A chain is called a *terminal chain* if it is incident to a leaf. It is clear that each major handle must be a major generalized handle.

For convenience, we define $T_{vu}(v)$ as the connected component of $T - vu$ containing v (normally it is call a *subtree of* u), and $T_u(uv)$ as the graph derived from adding $\{u, uv\}$ to $T_{vu}(v)$. Similarly, define $T_{[vu]}(v)$ and $T_u([uv])$ if $[vu]$ (or $[uv]$) is a chain of T. A subtree T' of T is called a *strong subtree* if there is no vertex $u \in V_3(T) \cap V_2(T')$. Clearly, $T_u([uv])$ is a strong subtree of T but $T_{[vu]}(v)$ is not.

From now on, we assume all trees T are of maximum degree 3. Let T' be a tree obtained from T by replacing each uv-chain as the edge uv. Thus the vertex set of T' is the set of leaves and major vertices of T.

Let $D(T)$ be a digraph with the same vertex set as T'. If $uv \in E(T')$ is the edge corresponding to the open chain $(uv]$ in T, then assign a direction from u to v. We keep the notation $(uv]$ to denote such an arc in $D(T)$. If $uv \in E(T')$ is the edge corresponding to the closed k-uv-chain, then duplicate it into two arcs: one is the arc from u to v and the other is from v to u. Keep the notation $[u(k)v]$ (or $[uv]$) and $[v(k)u]$ (or $[vu]$), which are called *out-arc* and *in-arc* of u, to denote such directed arcs in $D(T)$, respectively.

Next, assign non-negative weights to each major vertex of $D(T)$ (equivalently T) with respect to its incident in-arc. In order to distinguish weights and labels, we use circled numbers to represent the weights. Moreover, assume T contains at least two major vertices and so $D(T)$ is not isomorphic to $K_{1,3}$.

Weight Assignment:

Initial step: Every open oriented edge $(uv]$ gives weight ① to v.

Main procedure: If all vertices of $D(T)$ receive three weights, then Stop. Otherwise, consider each vertex u with neighborhood $\{v_1, v_2, v\}$ that has received at least two weights ⓐ and ⓑ from its in-arcs $[v_1u]$ and $[v_2u]$, respectively, and one of its out-arc $[u(k)v]$ has not assigned type. (The

existence of the vertex u will be proved after the following examples.) Assign the out-arc $[u(k)v]$ of u a type by means of Definition 1 stated below and give v a weight according to Table 1. Repeat this procedure.

After this assignment, each major vertex of T receives weights belong to $\{\textcircled{0}, \textcircled{1}, \textcircled{2}, \textcircled{3}, \textcircled{5}, \textcircled{6}, \textcircled{10}, \textcircled{15}\}$. Define the type of closed oriented chain as follows:

Definition 1 Let $[u(k)v]$ be a closed oriented chain and P_1, P_2 be the other oriented chains incident to u, where $k \geq 2$. Suppose P_1 and P_2 give weight $\textcircled{a}$ and $\textcircled{b}$ to u, respectively. If a and b are positive, then the chain $[u(k)v]$ is

(1) of type $(\textcircled{1}, k)$ if $a, b \in \{1, 2, 3, 5\}$ and $\gcd(a, b) = 1$;

(2) of type $(\textcircled{a}, k)$ if $a \in \{6, 10, 15\}$ and $\gcd(a, b) = 1$;

(3) of type $(\textcircled{d}, k)$ if $\gcd(a, b) = d \in \{2, 3, 5\}$;

(4) of type $(\textcircled{0}, k)$ if $\gcd(a, b) \in \{6, 10, 15\}$,

where $\gcd(a, b)$ is the greatest common divisor (gcd) of a and b. We also define $[u(k)v]$ of type $(\textcircled{0}, k)$ when one of a and b is zero.

Remark 2.1 Case (4) is equivalent to $a = b \in \{6, 10, 15\}$.

$[u(0)v]$ gives $\textcircled{6}$ to v; $[u(1)v]$ gives $\textcircled{15}$ to v.

Types of oriented $[u(k)v]$ $(k \geq 2)$							Weight assigned to v
$(\textcircled{1}, 2)$, $(\textcircled{1}, 4^+)$,	$(\textcircled{2}, 7^+)$,	$(\textcircled{3}, 7^+)$,	$(\textcircled{5}, 6^+)$,	$(\textcircled{6}, 5^+)$,	$(\textcircled{10}, 4^+)$,	$(\textcircled{15}, 5^+)$	$\textcircled{1}$
$(\textcircled{1}, 3)$,	$(\textcircled{2}, 5)$,	$(\textcircled{3}, 6)$,	$(\textcircled{5}, 3)$,	$(\textcircled{6}, 2)$,	$(\textcircled{10}, 3)$,	$(\textcircled{15}, 3)$	$\textcircled{2}$
	$(\textcircled{2}, 6)$,	$(\textcircled{3}, 5)$,			$(\textcircled{10}, 2)$,	$(\textcircled{15}, 4)$	$\textcircled{3}$
	$(\textcircled{2}, 4)$,		$(\textcircled{5}, 5)$,	$(\textcircled{6}, 4)$,		$(\textcircled{15}, 2)$	$\textcircled{5}$
	$(\textcircled{2}, 2)$,	$(\textcircled{3}, 3)$,	$(\textcircled{5}, 4)$,	$(\textcircled{6}, 3)$			$\textcircled{6}$
		$(\textcircled{3}, 2)$					$\textcircled{10}$
		$(\textcircled{3}, 4)$,	$(\textcircled{5}, 2)$				$\textcircled{15}$
$(\textcircled{0}, 2^+)$,	$(\textcircled{2}, 3)$						$\textcircled{0}$

k^+ means all the integers not less than k.

Table 1: The weight is given to the terminal v by $[u(k)v]$, $k \geq 0$.

Now we explain why the algorithm can assign all notes with three weights.

Let $r = \lceil \mathrm{diam}(T')/2 \rceil$ be the radius of T', where $\mathrm{diam}(T')$ is the diameter of T'. It is known that T' contains one or two centers depending on whether $\mathrm{diam}(T')$ is even or odd. Let c be a center of T' and consider T' as a rooted tree with root c.

If $r = 1$, then T' is $K_{1,3}$. Clearly, three weights can be assigned to the center. Hence we assume $r \geq 2$. In this case there are at least two pairs of leaves adjacent to their fathers, respectively. After the initial step, there are at least two vertices that receive two weights from their sons. So we may perform the main procedure. By removing all leaves of T', we denote the resulting tree by

T''. There are at least two pairs of leaves of T'' adjacent to their fathers respectively if the radius of T'' is still greater than 1. For those leaves of T'', they have already received two weights from their sons. By the same argument as before, there is at least two vertices receiving two weights. Therefore the assignment may continue and remove the leaves repeatedly until the radius of the last tree becomes 1. In this case, the last tree is either $K_{1,3}$ or P_2. Note that, up to now each vertex receives weights from its two sons. Also, after removing the leaves at each iteration, the height of T'' decreases exactly one. As a result, the last vertex(ices) receiving weights must be the center(s).

For the first case, the main procedure implies c receives three weights from its sons. For the second case, the main procedure implies c and the other center c' receive two weights from their sons. By the rule of the assignment, we perform the main procedure on c and c' and they receive weights from $[cc']$ and $[c'c]$, respectively.

Now we back to consider the original graph $D(T)$. If we remove all arcs that have types, the resulting graph is isomorphic to T''. At this stage, all centers receive three weights and other vertices receive two weights. Thus the main procedure can be performed from the center(s) to its(their) descendants step by step.

Remark 2.2 From the above assignment, it is easy to see that the weight given to a vertex u from its in-arc $[vu]$ is uniquely determined by the subtree of u containing the vertex v.

Definition 2 A strong subtree T^* of T with $\Delta(T) = 3$ is called *bad* if it satisfies the following conditions:

(1) $V_3(T^*) \neq \varnothing$.

(2) For each generalized major handle u of T^*, two of its terminal chains are closed 3-chains in T, or one of its incident chains is a closed 0-chain or 1-chain in T.

(3) There is no major vertex adjacent to another major vertex in T^* and no 2-vertex adjacent to two different major vertices in T^*.

(4) There is a vertex $u \in V_3(T^*)$ satisfying one of the following conditions

  (4.1) One of its incident closed chain $[uv]$ is type $(②, 3)$ in T.

  (4.2) Two of its incident chains give the same weight ⑥, ⑩ or ⑮ to u.

  (4.3) Vertex u receives three positive weights and the greatest common divisor of these weights is greater than 1.

  Such vertex u is called a *bad vertex*. A vertex is *good* if it is not bad.

A tree T is called a *bad tree* if it contains a bad strong subtree.

Example 2.1 Consider the following tree T. The middle figure is the corresponding tree T' and the right figure is the digraph $D(T)$. Hence T is a good tree. □

Example 2.2 Consider another tree T, we obtain $D(T)$ similarly. Vertex v receives the same weight $\textcircled{6}$. The gcd of three weights of vertex u received is $\textcircled{2}$. Closed chain $[uw]$ is of type $(\textcircled{2}, 3)$. Hence u and v are bad vertices and T is a bad tree. The following figure is a bad subtree of T.

In this tree, u is incident with a closed 0-chain and a closed 3-chain in T; v is incident with a closed 0-chain and an open 0-chain in T. $\square$

Remark 2.3 For any generalized major handle u, each terminal chain gives $\textcircled{1}$ to it. Then the closed chain, if any, is of type $(\textcircled{1}, k)$. Thus u does not satisfy (4) of Definition 2 and it is not bad vertex. In other words, a bad vertex is incident to at least two closed chains.

Remark 2.4 In a good tree, every oriented chain can give positive weight to its major terminal. Therefore, $\textcircled{0}$ weight can only be given to its major terminal in bad tree.

A configuration is called $< 333 >$ if it has a 3-vertex adjacent to two 3-vertices, or $< 32323 >$ if it has a 3-vertex adjacent two 2-vertices which is adjacent to another 3-vertex respectively (see Figure **??**). It is obvious that $< 333 >$ associates a bad subtree T^* which is the subtree induced by the vertex set $\{u_1, u_2, u, x\}$, because both two closed 0-chain give $\textcircled{6}$ to their common adjacent 3-vertex. Similarly, $< 32323 >$ associates a bad subtree T^* which is the subtree induced by the vertex set $\{u_1, u_2, y_1, y_2, u, x\}$.

Lemma 3 *Let T be a tree with $\Delta = 3$ and $[uv]$ be a closed chain, where u is a bad vertex. Let wu-chain P and $w'u$-chain Q be the other chains incident to u. Suppose that $[vu]$, P and Q give positive weights $\textcircled{a}$, $\textcircled{b}$ and $\textcircled{c}$ to u, respectively. Then one of the fo statements holds:*

(1) $b = c \in \{6, 10, 15\}$;

(2) $a \in \{6, 10, 15\}$ *and either* $a \in \{b, c\}$ *with* $\gcd(b, c) = 1$ *or* $\gcd(a, b, c) \in \{2, 3, 5\}$;

(3) $a \in \{2, 3, 5\}$ *and is a factor of* $\gcd(b, c)$.

Proof. Since $[vu]$, P and Q give weights to their ends, they are not of type $(\textcircled{2}, 3)$ according to Table 1. Moreover, by Remark 2.3, at least one of P and Q is closed.

Since u is a bad vertex and by Definition 2, we have three cases: (A) $[uv]$, $[uw]$ or $[uw']$ is of type $(\textcircled{2}, 3)$, or (B) two of a, b and c are same and in $\{6, 10, 15\}$, or (C) $\gcd(a, b, c) > 1$.

(A) By symmetry, we may assume that $[uv]$ is of type $(\textcircled{2}, 3)$. Then $\gcd(b, c) = 2$ by Definition 1. Since $[v(3)u]$ is not of type $(\textcircled{2}, 3)$ and $a > 0$, $a \in \{2, 6\}$ by Table 1. Thus, $\gcd(a, b, c) = 2$. It is referred to (C).

(B) If $b = c \in \{6, 10, 15\}$, then (1) holds. Suppose $a = b \in \{6, 10, 15\}$ or $a = c \in \{6, 10, 15\}$. Without loss of generality, we may assume $a = b \in \{6, 10, 15\}$ and $b \neq c$. Thus, $\gcd(a, b, c) \in \{1, 2, 3, 5\}$. If $\gcd(a, b, c) = 1$, then $\gcd(b, c) = 1$. Otherwise, $\gcd(a, b, c) \in \{2, 3, 5\}$ and we have (2).

(C) $\gcd(a,b,c) > 1$. Clearly $a \neq 1$.

Suppose $a \in \{2,3,5\}$. Since a is prime and $\gcd(a,b,c) > 1$, we have $\gcd(a,b,c) = a$ and so a is a factor of $\gcd(b,c)$, which implies (3).

Suppose $a \in \{6,10,15\}$. If $a \in \{b,c\}$, then it is referred to (B). Let $a \notin \{b,c\}$. If $b = c \in \{6,10,15\}$, then (1) holds. Otherwise, we have $\gcd(b,c) \in \{1,2,3,5\}$. Since $\gcd(a,b,c) > 1$, $\gcd(a,b,c) \in \{2,3,5\}$ and we have (2). □

Lemma 4 *Let T be a tree with $\Delta = 3$. Assume a closed chain $[u(k)v]$ and the other two chains incident to v give ⓐ, ⓓ and ⓓ′ to v respectively; $[v(k)u]$ and the other two chains incident to u give ⓑ, ⓒ and ⓒ′ to u respectively, where a, b, c, c', d, d' are positive and $k \geq 2$. Then u is bad if and only if v is bad. In other words, u is good if and only if v is good.*

Proof. By symmetry, assume u is bad. From Lemma 3 we know that $b \neq 1$. Hence, by Table 1, $k \leq 6$ and $[v(k)u]$ is one of the type listed at the second to seventh rows of Table 1.

Note that, $\gcd(c,c') \in \{6,10,15\}$ implies $c = c' \in \{6,10,15\}$ and so $a = 0$, which is not a case. Therefore $\gcd(c,c') \in \{1,2,3,5\}$.

Suppose $k = 2$.

(A1.1) $[v(2)u]$ is of type (②, 2). By Definition 1, $\gcd(d,d') = 2$ and from Table 1, $b = 6$.

By Lemma 3, without loss of generality, we have $c = 6$ with $\gcd(c,c') = 1$; or $\gcd(6,c,c') = 2$; or $\gcd(6,c,c') = 3$. From the note above and Definition 1, $[u(2)v]$ corresponds to type (⑥, 2); (②, 2) or (③, 2), respectively, which gives ②, ⑥ and ⑩ to v by Table 1. Thus $\gcd(a,d,d') = 2$ and so v is bad.

(A1.2) $[v(2)u]$ is of type (③, 2). By Definition 1, $\gcd(d,d') = 3$ and from Table 1, $b = 10$.

By Lemma 3, without loss of generality, we have $c = 10$ with $\gcd(c,c') = 1$; or $\gcd(10,c,c') = 2$; or $\gcd(10,c,c') = 5$. Similar to (A1.1), $[u(2)v]$ corresponds to type (⑩, 2); (②, 2) or (⑤, 2), respectively, which yields $a = 3, 6, 15$ by Table 1. Thus $\gcd(a,d,d') = 3$ and so v is bad.

(A1.3) $[v(2)u]$ is of type (⑤, 2). By Definition 1, $\gcd(d,d') = 5$ and from Table 1, $b = 15$.

By Lemma 3, without loss of generality, we have $c = 15$ with $\gcd(c,c') = 1$; or $\gcd(15,c,c') = 3$; or $\gcd(15,c,c') = 5$. Similarly, $[u(2)v]$ corresponds to type (⑮, 2); (③, 2) or (⑤, 2), respectively, which yields $a = 5, 10, 15$. Then $\gcd(a,d,d') = 5$ and hence v is bad.

(A1.4) $[v(2)u]$ is of type (⑥, 2). By Definition 1, without loss of generality, $d = 6$ with $\gcd(d,d') = 1$ and from Table 1, $b = 2$.

By Lemma 3, 2 is a divisor of $\gcd(c,c')$. From the note above, $\gcd(c,c') = 2$. Therefore, $[u(2)v]$ is of type (②, 2) and hence $a = 6$. Since both a and d are 6, v is bad.

(A1.5) $[v(2)u]$ is of type (⑩, 2). By Definition 1, without loss of generality, $d = 10$ with $\gcd(d,d') = 1$ and from Table 1, $b = 3$.

By Lemma 3, similar to the previous case, $\gcd(c,c') = 3$. Therefore, $[u(2)v]$ is of type (③, 2) and hence $a = 10$. Since both a and d are 10, v is bad.

(A1.6) $[v(2)u]$ is of type $(⑮, 2)$. By Definition 1, without loss of generality, $d = 15$ with $\gcd(d, d') = 1$ and from Table 1, $b = 5$.

By Lemma 3, similar to the previous case, $\gcd(c, c') = 5$. Therefore, $[u(2)v]$ is of type $(⑤, 2)$ and hence $a = 15$. Since both a and d are 15, v is bad.

Suppose $k = 3$.

(A2.1) $[v(3)u]$ is of type $(③, 3)$. By Definition 1, $\gcd(d, d') = 3$ and from Table 1, $b = 6$.

Similar to (A1.1), we have $[u(3)v]$ is of type $(⑥, 3)$, $(②, 3)$ or $(③, 3)$. Since $a > 0$, $[u(3)v]$ is not of type $(②, 3)$. For the other cases, $a = 6$ by Table 1. Then $\gcd(a, d, d') = 3$. Hence v is bad.

(A2.2) $[v(3)u]$ is of type $(⑤, 3)$ or $(⑩, 3)$. or $(⑮, 3)$ and from Table 1, $b = 2$.

Similar to (A1.4) we have $\gcd(c, c') = 2$. Therefore, $[u(3)v]$ is of type $(②, 3)$ and hence it is not a case.

(A2.3) $[v(3)u]$ is of type $(⑥, 3)$. By Definition 1, without loss of generality, $d = 6$ and $\gcd(d, d') = 1$ and from Table 1, $b = 6$.

Similar to (A2.1) we have $a = 6$, so v is bad as both a and d are 6.

It is similar for the cases $k = 4, 5, 6$ and the proofs are omitted. □

Lemma 5 *Let T and T' be good trees with $\Delta = 3$. Suppose that chains $[uv]$, P and Q give ⓐ, ⓑ and ⓒ to v respectively in T, and wv'-chain gives ⓓ to v' in T', where $d(w) = 1$ or 3 in T' and a, b, c, d are positive. Let T'' be the tree obtained from T by replacing the subtree of v containing u by the subtree of v' containing w.*

(1) *If d divides a, then T'' is also a good tree.*

(2) *If $a \in \{\gcd(d, b), \gcd(d, c)\}$ and $a \in \{2, 3, 5\}$, then T'' is also a good tree.*

Proof. Under the hypothesis and by the weighted assignment, we have the following figures: Suppose to the contrary that T'' is bad. Hence T'' has a strong subtree T^* satisfies the conditions in Definition 2. Suppose v (v') is not a vertex of T^*. The T^* lies in one of the components of $T'' - v$. Thus T^* is a strong subtree of either T or T', which contradicts with T and T' being good. As a result, v is a vertex of T^*.

Step 1: Vertex v is good in T^*.

Suppose vertex v is bad in T^*. By Lemma 3, there are four possible cases: (A) $b = c \in \{6, 10, 15\}$; (B) $d \in \{6, 10, 15\}$, $d \in \{b, c\}$ and $\gcd(b, c) = 1$; (C) $d \in \{6, 10, 15\}$ and $\gcd(d, b, c) \in \{2, 3, 5\}$; (D) $d \in \{2, 3, 5\}$ and d is a factor of $\gcd(b, c)$.

(1) d divides a.

Since v is good in T, $\gcd(a,b,c) = 1$. Moreover, if $b = c$, then $b \notin \{6, 10, 15\}$. The former property implies that $\gcd(d,b,c) = 1$. Combining these two properties, only (B) will occur, i.e., $d \in \{6, 10, 15\}$, $d \in \{b, c\}$ and $\gcd(b,c) = 1$. Without loss of generality, we may assume $b = d$. Since d divides a, we have $a = b = d$. This implies that v is bad in T and contradiction occurs.

(2) $a \in \{\gcd(d,b), \gcd(d,c)\}$ and $a \in \{2, 3, 5\}$.

Without loss of generality, we assume $\gcd(d,b) = a \in \{2, 3, 5\}$. Since v is good in T, $\gcd(a,b,c) = 1$. Since a is a prime factor of b, $\gcd(a,c) = 1$. This implies that $\gcd(d,b,c) = \gcd(a,c) = 1$ and so only (B) or (D) will occur.

Suppose (B) holds. Since d is composite and $\gcd(d,b) = a$, which is a prime, so $d \neq b$. Hence $d = c$. This contradicts with $\gcd(b,c) = 1 \neq a$.

Suppose (D) holds. Since a and d are prime, by our assumption $a = d$. Thus, $\gcd(a,b,c) = 1$ implies that it is impossible for d being a factor of $\gcd(b,c)$.

Step 2: Vertex w is good in T^*.

We only need to consider when u is a major vertex. Since T' is a good tree, w receives two positive weights from T'. Also, T^* is bad subtree satisfying conditions of Definition 2 implies that there are no closed 0-chain and 1-chain in T^*. Suppose $[vw]$ sends weight $\textcircled{0}$ to w. Since we have proved that v is good in T^*, $[vw]$ is only possible of type $(\textcircled{0}, k)$ for some $k \geq 2$. By Remark 2.1 we have $b = c \in \{6, 10, 15\}$, which contradicts with v being good in T^*. Since v is good, so w is good in T^* by Lemma 4.

Step 3: Suppose P and/or Q are closed chains with other ends x and y, respectively. Similar to Step 2, x and y are good in T^*.

Consider T^* as a rooted tree with root v. By the same proof as above, we can prove that the major descendants of v are good. Thus T^* is a good tree, which yields a contradiction. $\square$

Lemma 6 *If T is a good tree with $\Delta = 3$, then any strong subtree of T is also good.*

Proof. Let S be a strong subtree of T with vertices $x_1, \ldots, x_s$ that are k-vertices in T but leaves in S, where $k = 2, 3$. Then S is obtained from T by suitably removing $k-1$ subtrees of each x_i. In order to prove this lemma, it suffices to prove that a strong subtree obtained from T by removing $k-1$ subtrees of a k-vertex x is still good, where $k = 2, 3$.

Let v be the nearest major vertex apart from x in S. Let T' be the tree consisting of the chain $(xv]$ by adding two leaves to v. Clearly T' is good. Now S is the tree obtained from T by replacing the subtree of v containing x by the subtree of v in T' containing x. By substituting $d = 1$ in Lemma 5(1), we conclude that S is good.

$\square$

Lemma 7 *Let T be a tree with $\Delta = 3$ and $V_3(T) \geq 3$. If T does not contain $< 333 >$ and $< 32323 >$, then T contains at least one of the following configurations (see Figure* **??** *and* **??** *):*

(C1) *A path* $ux_1x_2\cdots x_7$ *where* u *is a* 3*-vertex and* $x_1, x_2, \ldots, x_7$ *are* 2*-vertices;*

(C2) *a leaf* u *adjacent to a* 2*-vertex* v*;*

(C3) *a* 3*-vertex* v *incident to a closed* 0*-chain* $[u(0)v]$ *and an open* 0*-chain* $(y(0)v]$*, where* u *is a major handle;*

(C4) *a closed* k*-chain* $[u(k)v]$*, where* u *is a major handle and*

- (C4.1) $k = 2$*;*
- (C4.2) $k \in \{4, 5, 6\}$*;*

(C5) *a* 3*-vertex* v *incident to two chains: a closed* 3*-chain* $[u(3)v]$*, where* u *is a major handle and*

- (C5.1) *an open* 0*-chain* $(y(0)v]$*;*
- (C5.2) *a closed* 0*-chain* $[y(0)v]$*, where* y *is a major handle;*
- (C5.3) *a closed* 1*-chain* $[y(1)v]$*, where* y *is a major handle;*
- (C5.4) *a closed* 3*-chain* $[y(3)v]$*, where* y *is a major handle;*

(C6) *a* 3*-vertex* v *incident to three chains: a closed* 1*-chain* $[u(1)v]$*, where* u *is a major handle; an open* 0*-chain* $(y(0)v]$*; and a closed* k*-chain* $[v(k)w]$*, where*

- (C6.1) $k = 0$*;*
- (C6.2) $k \in \{3, 5, 6\}$*;*

(C7) *a* 3*-vertex* v *incident to three chains: a closed* 1*-chain* $[u(1)v]$*, where* u *is a major handle; an open 0-chain* $(y(0)v]$*; and a closed* 2*-chain* $[v(2)w]$*, while* w *is incident to*

- (C7.1) *an open* 0*-chain* $(w'(0)w]$*;*
- (C7.2) *a closed* 0*-chain* $[w'(0)w]$*, where* w' *is a major handle;*
- (C7.3) *a closed* 1*-chain* $[w'(1)w]$*, where* w' *is a major handle;*
- (C7.4) *a closed* 3*-chain* $[w'(3)w]$*, where* w' *is a major handle;*
- (C7.5) *a closed* 2*-chain* $[w'(2)w]$ *and* w' *is incident to an open* 0*-chain* $(w_1(0)w']$ *and a closed* 1*-chain* $[w_2(1)w']$*, where* w_2 *is a major handle;*

(C8) *a* 3*-vertex* v *incident to three chains: a closed* 1*-chain* $[u(1)v]$*, where* u *is a major handle; a closed* 0*-chain* $[u'(0)v]$*, where* u' *is a major handle; and a closed* 2*-chain* $[v(2)w]$*, while* w *is incident to*

- (C8.1) *an open* 0*-chain* $(w'(0)w]$*;*
- (C8.2) *a closed* 0*-chain* $[w'(0)w]$*, where* w' *is a major handle;*
- (C8.3) *a closed* 1*-chain* $[w'(1)w]$*, where* w' *is a major handle;*
- (C8.4) *a closed* 3*-chain* $[w'(3)w]$*, where* w' *is a major handle;*

(C8.5) *a closed* 2*-chain* $[w'(2)w]$ *and* w' *is incident to an open* 0*-chain* $(w_1(0)w']$ *and a closed* 1*-chain* $[w_2(1)w']$*, where* w_2 *is a major handle;*

(C8.6) *a closed* 2*-chain* $[w'(2)w]$ *and* w' *is incident to a closed* 0*-chain* $[w_1(0)w']$ *and a closed* 1*-chain* $[w_2(1)w']$*, where* w_1 *and* w_2 *are major handles;*

(C9) *a* 3*-vertex* v *incident to three chains: a closed* 1*-chain* $[u(1)v]$*, where* u *is a major handle; an open* 0*-chain* $(y(0)v]$*; and a closed* 4*-chain* $[v(4)w]$*, while* w *is incident to*

(C9.1) *an open* 0*-chain* $(w'(0)w]$*;*

(C9.2) *a closed* 0*-chain* $[w'(0)w]$*, where* w' *is a major handle;*

(C9.3) *a closed* 1*-chain* $[w'(1)w]$*, where* w' *is a major handle;*

(C9.4) *a closed* 3*-chain* $[w'(3)w]$*, where* w' *is a major handle;*

(C9.5) *a closed* 2*-chain* $[w'(2)w]$ *and* w' *is incident to an open* 0*-chain* $(w_1(0)w']$ *and a closed* 1*-chain* $[w_2(1)w']$*, where* w_2 *is a major handle;*

(C9.6) *a closed* 2*-chain* $[w'(2)w]$ *and* w' *is incident to a closed* 0*-chain* $[w_1(0)w']$ *and a closed* 1*-chain* $[w_2(1)w']$*, where* w_1 *and* w_2 *are major handles;*

(C9.7) *a closed* 4*-chain* $[w'(4)w]$ *and* w' *is incident to an open* 0*-chain* $(w_1(0)w']$ *and a closed* 1*-chain* $[w_2(1)w']$*, where* w_2 *is a major handle;*

(C10) *a* 3*-vertex* v *incident to three chains: a closed* 1*-chain* $[u(1)v]$*, where* u *is a major handle; a closed* 0*-chain* $[u'(0)v]$*, where* u' *is a major handle; and a closed* k*-chain* $[v(k)w]$*, where* $k \in \{3, 4, 5, 6\}$*.*

The proof of Lemma 7 is not difficult but tedious only. The main idea of the proof of Lemma 7 is to consider every vertex of a longest path of a tree, similar to those in [11] and [13]. The remaining parts are only careful and tedious analysis and we omit the proof here.

3 Main results

Theorem 8 ([6]) *For every tree* T*,* $\Delta + 1 \leq \lambda(T) \leq \Delta + 2$*.*

Lemma 9 ([11]) *If* T *is a tree with* $\Delta = 3$ *and* f *is a* 4*-*$L(2,1)$*-labeling of* T*, then* $f(u) = 0$ *or* 4 *for every major vertex* u*.*

Theorem 10 *Let* T *be a tree with* $\Delta = 3$*. Then* $\lambda(T) = 4$ *if and only if* T *is good.*

We give the proof of Theorem 10 by considering the sufficiency and necessity of T is good.

3.1 Sufficiency

Theorem 8 shows that $\lambda(T) \geq 4$ for any tree T with $\Delta = 3$. Hence in this subsection we assume that T is a good tree. It suffices to show that T has a 4-$L(2,1)$-labeling. It is easy to obtain a 4-$L(2,1)$-labeling if $|V_3(T)| \leq 2$, therefore we assume that $|V_3(T)| \geq 3$.

Remark 3.1 Suppose that a tree T with $\Delta = 3$ has a 4-$L(2,1)$-labeling f using the label set $\mathcal{B} = \{0,1,2,3,4\}$. Define $f' = 4 - f$, then f' is also a 4-$L(2,1)$-labeling of T, which is called *the symmetric labeling of f*.

We prove that T has a 4-$L(2,1)$-labeling using $\mathcal{B}$ by induction on $|T|$. Since T is good, T does not contain the configurations $< 333 >$ and $< 32323 >$. By Lemma 7, we only need to deal with cases (C1)-(C10).

(C1) There is a path $ux_1x_2 \cdots x_7$, where u is a 3-vertex and $x_1, x_2, \ldots, x_7$ are 2-vertices. Assume v is the another neighbor of x_7 besides x_6.

Let $T' = T \setminus \{x_2, x_3, \ldots, x_6\}$. Then T' consists of two components, say T_1 and T_2. Assume that $u \in V(T_1)$ and $v \in V(T_2)$. Since $V_3(T) \cap V_2(T') = \varnothing$, T_i are strong subtrees of T, where $i = 1, 2$. By Lemma 6, they are good. Thus, T' has a 4-$L(2,1)$-labeling f by induction hypothesis. By Remark 3.1 and Lemma 9 we may assume that $f(u) = 0$. In this case $f(x_1) \in \{2,3,4\}$.

(1.1) Suppose that $f(x_1) = 2$. By Remark 3.1 we may assume that $f(v) \in \{0,1,2\}$. Note that the label of x_7 has some restrictions depended on the label of v. For example, when $f(v) = 1$ then $f(x_7) \in \{3,4\}$. Following is the label assignment for x_2, x_3, x_4, x_5, x_6:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	v
2	4	1	3	0	4	2	0
2	4	0	2	4	1	3	0
2	4	1	3	0	2	4	0
2	4	0	2	4	0	3	1
2	4	1	3	0	2	4	1
2	4	0	3	1	4	0	2

(1.2) Suppose that $f(x_1) = 3$. In order to have a 4=$L(2,1)$-labeling, the label of x_2 and x_3 must be 1 and 4, respectively. By Remark 3.1 we may assume that $f(v) \in \{0,3,2\}$. Following is the label assignment for x_2, x_3, x_4, x_5, x_6 except when $f(v) = 0$ and $f(x_7) = 3$:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	v
3	1	4	2	0	4	2	0
3	1	4	0	3	1	4	0
3	1	4	2	0	4	1	3
3	1	4	0	2	4	0	3
3	1	4	0	3	1	4	2

For the case $f(x_7) = 3$ and $f(v) = 0$, we relabel T_2 by the symmetric labeling of f. Hence the labels of x_7 and v are 1 and 4, respectively. Then we label x_2, x_3, x_4, x_5, x_6 by $1, 4, 2, 0, 3,$ respectively.

(1.3) Suppose that $f(x_1) = 4$. Similar to the case above by relabeling T_2 if necessary, we

have the following assignment x_2, x_3, x_4, x_5, x_6:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	v
4	2	0	3	1	4	2	0
4	1	3	0	4	1	3	0
4	2	0	4	1	3	0	4
4	1	3	0	2	4	1	3
4	2	0	4	2	0	4	1
4	2	0	4	2	0	4	2

As a result, we only consider T contains closed k-chain, where $k \leq 6$, in the remaining cases.

(C2) There is a leaf u adjacent to a 2-vertex v. Let w be the other neighbor of v. Let $T' = T - u$, then T' has a 4-$L(2,1)$-labeling f by Lemma 6 and induction hypothesis. We may label u by the element in $\mathcal{B} \setminus \{f(w), f(v), f(v)-1, f(v)+1\}$.

(C3) There is a 3-vertex v incident to a closed 0-chain $[u(0)v]$ and an open 0-chain $(y(0)v]$ such that u is a major handle. Let y_1 and y_2 be the leaves adjacent to the handle u. Since $|V_3(T)| \geq 3$, v must be incident to another closed k-chain $[v(k)w] = vx_1x_2 \cdots x_kw$. From Case (C1) and because T does not contain the configuration $< 333 >$, so $1 \leq k \leq 6$ in the remaining parts of this proof. Note that, $[u(0)v]$ gives ⑥ to v and $(y(0)v]$ gives ① to v, so $[v(k)w]$ is type (⑥, k). We consider the following cases with different values of k:

(3.1) $k = 1$. Let $T' = T - \{y_1, y_2\}$. Then T' has a 4-$L(2,1)$-labeling f by Lemma 6 and induction hypothesis. By Lemma 9 and Remark 3.1, we may assume $f(v) = 0$ and hence $f(w) = 4$ and $f(x_1) = 2$. Then relabel y by 3 and u by 4. Finally, set $f(y_1) = 1$ and $f(y_2) = 2$ in T.

(3.2) $k = 2$. Then $[v(2)w]$ is of type (⑥, 2) and gives ② to w. We construct a new tree $T' = T_{wx_2}(w) + [w(3)z_1] + [z_1(0)z_2) + [z_1(0)z_3)$ and $z_0 \in [w(3)z_1]$ be the neighbor of w. Note that it is isomorphic to $T - y$. It is easy to see that $[z_1(3)w]$ also gives ② to w. Then T' is a good tree and has a 4-$L(2,1)$-labeling f by Lemma 5 and induction hypothesis. We may assume $f(w) = 0$ and $f(z_0) = \alpha$. Hence $\alpha \in \{3, 4\}$, otherwise z_1 cannot be labeled under f by Lemma 9. Now we label x_2 by α, i.e., either 3 or 4. Hence, assign proper label sequence 03140 or 04204 to wx_2x_1vu in T. Thus, f can be extended to T after labeling the leaves adjacent to u and v easily since $f(u), f(v) \in \{0, 4\}$.

(3.3) $k = 3$. Then $[v(3)w]$ is of type (⑥, 3) and gives ⑥ to w. Let $T' = T_{wx_3}(w) + [w(0)z_1] + [z_1(0)z_2) + [z_1(0)z_3)$. It is easy to see that $[z_1(0)w]$ gives ⑥ to w. Therefore, T' is a good tree by Lemma 5 and has a 4-$L(2,1)$-labeling f by induction hypothesis. By Lemma 9, we may assume $f(w) = 0$ and $f(z_1) = 4$. Hence, assign proper label sequence 041304 to $wx_3x_2x_1vu$ in T. Thus, f can be extended to T after labeling the leaves adjacent to u and v.

(3.4) $k = 4$. Then $[v(4)w]$ is of type (⑥, 4) and gives ⑤ to w. Let $T' = T_{wx_4}(w) + [w(2)z_1] + [z_1(0)z_2) + [z_1(1)z_3] + [z_3(0)z_4) + [z_3(0)z_5)$ and $z_0 \in [w(2)z_1]$ be the neighbor of w. It

is easy to see that $[z_1(2)w]$ is of type $(⑮, 2)$ and gives ⑤ to w. Therefore, T' is a good tree by Lemma 5 and has a 4-$L(2,1)$-labeling f by induction hypothesis. It is clear that $f(w), f(z_1), f(z_3) \in \{0,4\}$ by Lemma 9. If $f(z_0) \in \{0,4\}$, then the vertex adjacent with z_1 and z_3 and the vertex adjacent with z_1 and w must be labeled by 2 which is impossible. So we get $f(z_0) \notin \{0,4\}$. Assume $f(w) = 0$. We label x_4 by $f(z_0) \in \{2,3\}$. The possible label sequence of the path $wx_4x_3x_2x_1vu$ is 0241304 or 0314204 in T. Thus, f can be extended to T after labeling the leaves adjacent to u and v.

(3.5) $k = 5$. Let $T' = T_{x_5x_4}(x_5)$. So $T' \subset T$ is a good tree by Lemma 6. By induction hypothesis, T' has a 4-$L(2,1)$-labeling f with $f(w) = 0$. Then $f(x_5) = 2$, 3, or 4. According to these cases, we label $x_5x_4x_3x_2x_1vu$ in T with sequence 2403140, 3140240, or 4130240, respectively.

(3.6) $k = 6$. Similar to the above case, $T' = T_{x_5x_4}(x_5)$ has a 4-$L(2,1)$-labeling f with $f(w) = 4$ by induction hypothesis.

If $f(x_6) = 0$, then $f(x_5) = 2$, 3 or 4 which is the same as (3.5).

If $f(x_6) = 1$, then and $f(x_5) = 3$ or 4. We label the path $x_5x_4x_3x_2x_1vu$ in T with label sequence 3041304 or 4204204 accordingly.

If $f(x_6) = 2$, then $f(x_5) = 0$. We label the path $x_5x_4x_3x_2x_1vu$ in T with label sequence 0314204.

Thus, f can be extended to T after labeling the leaves adjacent to u and v.

(C4) There is a 3-vertex v incident to a closed k-chain $[u(k)v] = ux_1x_2 \cdots x_kv$ such that u is a major handle, where $k \in \{2,4,5,6\}$.

(4.1) $k = 2$. Let $T' = T_{x_2x_1}(x_2)$. Then $T' \subset T$ is a good tree by Lemma 6. By the induction hypothesis, T' has a 4-$L(2,1)$-labeling f with $f(v) = 0$. Whatever the label of x_2 is in T', we always can assign proper label sequence 0240, 0314 or 0420 to the path vx_2x_1u in T. Thus, f can be extended to T after labeling the leaves adjacent to u.

(4.2) $k \in \{4,5,6\}$. Let $T' = T_{x_4x_3}(x_4)$. Then $T' \subset T$ is a good tree by Lemma 6. By the induction hypothesis, T' has a 4-$L(2,1)$-labeling f. By Remark 3.1 we may assume that $f(x_5) \in \{0,1,2\}$ when $k = 5,6$ (or $f(v) = 0$ when $k = 4$).

If $f(x_5) = 0$ (or $f(v) = 0$ when $k = 4$), then $f(x_4) = 2$, 3 or 4. We label the path $x_4x_3x_2x_1u$ in T with label sequence 24130, 31420 or 41304 accordingly.

If $f(x_5) = 1$, then $f(x_4) = 3$ or 4. We label the path $x_4x_3x_2x_1u$ in T with label sequence 30420 or 40314 accordingly.

If $f(x_5) = 2$, then $f(x_4) = 0$ or 4. We label the path $x_4x_3x_2x_1u$ in T with label sequence 03140 or 41304 accordingly.

Thus, f can be extended to T after labeling the leaves adjacent to u.

(C5) There is a 3-vertex v incident to two chains: one is a closed 3-chain $[u(3)v] = ux_1x_2x_3v$ such that u is a major handle. The other chain Q is an open 0-chain $(y(0)v]$ or a closed chain $[y(k)v]$ with $k = 0,1,3$ such that y is a major handle.

(5.1) $Q = (y(0)v]$. Let $T' = T_{x_3x_2}(x_3)$. By the induction hypothesis, T' has a 4-L(2,1)-labeling f with $f(v) = 0$. If $f(x_3) = 2$, exchange the labels of y and x_3. Hence, $f(x_3) = 3$ or 4. We can assign 03140 or 04204 to $vx_3x_2x_1u$ in T. Thus, f can be extended to T after labeling the leaves adjacent to u.

(5.2) $Q = [y(0)v]$ such that y is a major handle. Let P be the chain incident to v besides $[u(3)v]$ and $[y(0)v]$. If P is open, then we can label all vertices of T easily. Assume $P = uy_1 \cdots y_k w$ is closed. Since T does not contain $< 333 >$ and by Case (C1), $1 \le k \le 6$. Note that, $[u(3)v]$ and $[y(0)v]$ give ② and ⑥ to v respectively, so $[v(k)w]$ is of type $(②, k)$. Since T is good, $k \neq 3$. Next we consider the following cases depending on the values of k:

(5.2-1) $k = 1$. Let $T' = T_{x_3x_2}(x_3)$. Then T' has a 4-$L(2,1)$-labeling f with $f(v) = 0$ by Lemma 6 and the induction hypothesis. Then $f(y_1) = 2$ and $f(y) = 4$ and it deduces $f(x_3) = 3$ in T. Assign proper label sequence 03140 to $vx_3x_2x_1u$. Thus, f can be extended to T after labeling the leaves adjacent to u and y.

(5.2-2) $k = 2$. Then $[v(2)w]$ is of type $(②, 2)$ and gives ⑥ to w. Let $T' = T_{wy_2}(w) + [w(0)z_1] + [z_1(0)z_2) + [z_1(0)z_3)$. Same as (3.3) we have $[z_1(0)w]$ gives ⑥ to w, T' is a good tree and has a 4-$L(2,1)$-labeling f with $f(w) = 0$ and $f(z_1) = 4$. We assign proper label sequence 04203140 to $wy_2y_1vx_3x_2x_1u$ and 4 to y. Thus, f can be extended to T after labeling the leaves adjacent to u and y.

(5.2-3) $k = 4$. Then $[v(4)w]$ is of type $(②, 4)$ and gives ⑤ to w. Let $T' = T_{wy_4}(w) + [w(2)z_1] + [z_1(0)z_2) + [z_1(1)z_3] + [z_3(0)z_4) + [z_3(0)z_5)$ and $z_0 \in [w(2)z_1]$ be the neighbor of w. Same as (3.4) we have $[z_1(2)w]$ gives ⑤ to w, T' is a good tree and has a 4-$L(2,1)$-labeling f with $f(w) = 0$ and $f(z_0) \in \{2, 3\}$. We assign proper label sequence 0240241304 or 0314203140 to $wy_4y_3y_2y_1vx_3x_2x_1u$ in T and label y with 0 or 4, respectively. Thus, f can be extended to T after labeling the leaves adjacent to u and y.

(5.2-4) $k = 5$. Then $[v(5)w]$ is of type $(②, 5)$ and gives ② to w. Let $T' = T_{wy_5}(w) + [w(3)z_1] + [z_1(0)z_2) + [z_1(0)z_3)$ and $z_0 \in [w(3)z_1]$ be the neighbor of w. Same as (3.2) we have $[z_1(3)w]$ gives ② to w, T' is a good tree and has a 4-$L(2,1)$-labeling f with $f(w) = 0$ and $f(z_0) = 3$ or 4. Hence, we assign proper label sequence 0314024 or 0413024 to $wy_5y_4y_3y_2y_1v$ and then let $f(y) = 0$ and assign 41304 to $vx_3x_2x_1u$ in T. Thus f can be extended to T after labeling the leaves adjacent to u and y.

(5.2-5) $k = 6$. Then $[v(6)w]$ is of type $(②, 6)$ and gives ③ to w. Let $T' = T_{wy_6}(w) + [w(4)z_1] + [z_1(0)z_2) + [z_1(1)z_3] + [z_3(0)z_4) + [z_3(0)z_5)$ and $z_0 \in [w(4)z_1]$ be the neighbor of w. It is easy to see that $[z_1(4)w]$ is of type by $(⑮, 4)$ and gives ③ to w. So T' is a good tree by Lemma 5 and has a 4-$L(2,1)$-labeling f with $f(w) = 0$ by induction hypothesis. Since $f(z_1), f(z_3) \in \{0, 4\}$, we can check that $f(z_0) \in \{2, 4\}$. We assign proper label sequence 024130241304 or 041304203140 to $wy_6 \cdots y_1vx_3x_2x_1u$ and label y with 0 or 4 in T, respectively. Thus, f can be extended to T after labeling the leaves adjacent to u and y.

(5.3) $Q = [y(1)v]$ such that y is a major handle. Let $T' = T_{x_3x_2}(x_3)$. Then T' has a 4-$L(2,1)$-labeling f with $f(v) = 0$ by the induction hypothesis. Hence $f(y) = 4$. Let z be the common neighbor of v and y. Then $f(z) = 2$ and so $f(x_3) = 3$ or 4. We assign proper sequence 03140 or 04204 to $vx_3x_2x_1u$, accordingly. Thus, f can be extended to T after labeling the leaves adjacent to u.

(5.4) $Q = [y(3)v]$ such that y is a major handle. Let $Q = yy_1y_2y_3v$. Note that both $[u(3)v]$ and $[y(3)v]$ give ② to v. Let $T' = T_{vy_3}(v) + [v(0)z_1] + [z_1(0)z_2) + [z_1(0)z_3)$. It is easy to see that $[z_1(0)v]$ gives ⑥ to v. By the Lemma 5 and the induction hypothesis, T' is a good tree and it has a 4-$L(2,1)$-labeling f with $f(v) = 0$. Hence $f(z_1) = 4$ and so we properly label y_3 with 4 in T. Next, we assign proper label sequence 04204 to $vy_3y_2y_1y$. Finally, f can be extended to T after labeling the leaves adjacent to y.

(C6) There is a 3-vertex v incident to three chains: a closed 1-chain $[u(1)v] = uxv$ such that u is a major handle; an open 0-chain $(y(0)v]$; and a closed k-chain $[v(k)w] = vy_1y_2\cdots y_kw$, where $k \in \{0, 3, 5, 6\}$.

(6.1) $k = 0$.
Let $T' = T_{xu}(x)$. Then $T' \subset T$ is a good tree by Lemma 6. By induction hypothesis, T' has a 4-$L(2,1)$-labeling f with $f(v) = 0$. Hence $f(w) = 4$. Relabel y by 3 and x by 2. Finally, label u by 4 in T. Thus, f can be extended to T after labeling the leaves adjacent to u.

(6.2) $k = 3$. Let $T' = T_{xu}(x)$. Then $T' \subset T$ is a good tree by Lemma 6 and has 4-$L(2,1)$-labeling f with $f(v) = 0$ by induction hypothesis. It is easy to see that $f(y_1) \neq 2$ since $f(v), f(w) \in \{0, 4\}$. Then label x with 2. Finally, label u with 4 in T. Thus, f can be extended to T after labeling the leaves adjacent to u and v.

(6.3) $k = 5$ or 6. Let $T' = T_{y_ky_{k-1}}(y_k)$. Then $T' \subset T$ is a good tree by Lemma 6 and has 4-$L(2,1)$-labeling f with $f(w) = 0$ by induction hypothesis. No matter $f(x_k) = 2$, 3 or 4, we always assign proper label sequence to $wy_k \cdots y_1v$ in T by 0240240, 0314204 or 0420314 when $k = 5$; assign 02403140, 03140314 or 04130240 when $k = 6$. For each case, $f(v) \in \{0, 4\}$ and $f(y_1) \neq 2$. As a result, we can set $f(x) = 2$ and $f(u) = 4 - f(v)$. Thus, f can be extended to T after labeling the leaves adjacent to u and v.

(C7) There is a 3-vertex v incident to three chains: a closed 1-chain $[u(1)v] = uu_1v$ where u is a major handle; an open 0-chain $(y(0)v]$; and a closed 2-chain $[v(2)w] = vx_1x_2w$.

(7.1) w is incident to an open 0-chain $(w'(0)w]$. Let $T' = T_{x_2x_1}(x_2)$. Then $T' \subset T$ is a good tree by Lemma 6 and has 4-$L(2,1)$-labeling f with $f(w) = 0$. If $f(x_2) = 4$, then exchange the labels of w' and x_2. As a result, $f(x_2) \in \{2, 3\}$. Assign proper label sequence 024024 or 031402 to $wx_2x_1vu_1u$ accordingly. Thus, f can be easily extended to T.

(7.2) w is incident to a closed 0-chain $[w'(0)w]$ such that w' is a major handle. Let $T' = T_{x_2x_1}(x_2)$. Then T' is a good tree and hence has 4-$L(2,1)$-labeling f with $f(w) = 0$ by induction hypothesis. Thus, $f(w') = 4$. Similar to the (7.1), f can be extended to T.

(7.3) w is incident to a closed 1-chain $[w'(1)w]$ such that w' is a major handle.

Let $[w(k)w''] = wy_1y_2\cdots y_kw''$ be the chain incident to w besides $[v(2)w]$ and $[w'(1)w]$. Since T does not contain $< 32323 >$, $k \neq 1$ and thus $k \in \{0, 2, 3, 4, 5, 6\}$. Note that, $[v(2)w]$ is of type $(⑮, 2)$ and gives ⑤ to w; while $[w'(1)w]$ gives ⑮ to w. Hence $[w(k)w'']$ is of type $(⑤, k)$. We consider the following cases with different values of k:

(7.3-1) $k = 0$. Let $T' = T_{x_2x_1}(x_2)$. Then T' is a good tree and hence has 4-$L(2,1)$-labeling f with $f(w) = 0$ by induction hypothesis. Then $f(w'') = 4$ and $f(w') = 4$, which induces $f(x_2) = 3$. Assign proper label sequence 1420 to x_1vu_1u. Thus, f can be extended to T.

(7.3-2) $k = 2$. Note that $[w(2)w'']$ is of type $(⑤, 2)$ and gives ⑮ to w''. Let $T' = T_{w''y_2}(w'') + [w''(1)z_1] + [z_1(0)z_2) + [z_1(0)z_3)$, where $[w''(1)z_1] = w''z_0z_1$. It is easy to see that $[z_1(1)w'']$ gives ⑮ to w'' too. Therefore, T' is a good tree and has 4-$L(2,1)$-labeling f with $f(w'') = 0$ by Lemma 5 and induction hypothesis. Then $f(z_1) = 4$ and $f(z_0) = 2$. Hence, assign proper label sequence 0240 to $w''y_2y_1w$ in T. Note that $f(y_1) = 4$ and so, similar to (7.3-1), f can be extended to T.

(7.3-3) $k = 3$. Note that $[w(3)w'']$ is of type $(⑤, 3)$ and gives ② to w''. Let $T' = T_{w''y_3}(w'') + [w''(3)z_1] + [z_1(0)z_2) + [z_1(0)z_3)$ and $z_0 \in [w''(3)z_1]$ be the neighbor of w''. Same as (3.2) we have $[z_1(3)w'']$ gives ② to w'', T' is a good tree and has a 4-$L(2,1)$-labeling f with $f(w'') = 0$ and $f(z_0) = 3$ or 4. So we may label y_3 by $f(z_0)$. And assign proper label sequence 3140 or 4204 to $y_3y_2y_1w$ in T according to $f(z_0) = 3$ or 4. Note that $f(y_1) \in \{0, 4\}$ and so, similar to (7.3-1), f can be extended to T.

(7.3-4) $k = 4$. Note that $[w(4)w'']$ is of type $(⑤, 4)$ and gives ⑥ to w''. Let $T' = T_{w''y_4}(w'') + [w''(0)z_1] + [z_1(0)z_2) + [z_1(0)z_3)$. Same as (3.3) we have $[z_1(0)w'']$ gives ⑥ to w'', T' is a good tree and has a 4-$L(2,1)$-labeling f with $f(w'') = 0$ and $f(z_1) = 4$. Then we can assign proper sequence 41304 to $y_4y_3y_2y_1w$ in T. Note that $f(y_1) = 0$ and so, similar to (7.3-3), f can be extended to T.

(7.3-5) $k = 5$. Note that $[w(5)w'']$ is of type $(⑤, 4)$ and gives ⑤ to w''. Let $T' = T_{y_5w''}(w'') + [w''(2)z_1] + [z_1(0)z_2) + [z_1(1)z_3] + [z_3(0)z_4) + [z_3(0)z_5)$ and $z_0 \in [w''(2)z_1]$ be the neighbor of w''. Same as (3.4) we have $[z_1(2)w'']$ gives ⑤ to w'', T' is a good tree and has a 4-$L(2,1)$-labeling f with $f(w'') = 0$ and $f(z_0) \in \{2, 3\}$. Then we assign proper label sequence 241304 or 314204 to $y_5\cdots y_1w$ in T according to $f(z_0) = 2$ or 3. Note that $f(y_1) = 0$ and so, similar to (7.3-3), f can be extended to T.

(7.3-6) $k = 6$. Let $T' = T_{y_6y_5}(y_6)$. Then T' is a good tree and hence has a 4-$L(2,1)$-labeling f with $f(w'') = 0$ by induction hypothesis. Whatever the label of y_6 is, assign proper label sequence 2403140, 3140240 or 4130240 to $y_6\cdots y_1w$ in T. Therefore, $f(y_1) = 4$ and hence f, similar to (7.3-1), f can be extended to T.

(7.4) w is incident to a closed 3-chain $[w'(3)w] = w'y_1y_2y_3w$, where w' is a major handle.

Let $T' = T_{y_3y_2}(y_3)$. Then T' is a good tree and hence has a 4-$L(2,1)$-labeling f with $f(w) = 0$ by induction hypothesis. Since $f(u), f(v) \in \{0,4\}$ and $f(w) = 0$, we may show that $f(x_2) = 2$ or 3. If $f(y_3) = 2$, then $f(x_2) = 3$. Exchange the labels of x_2 and y_3. Next, relabel x_1vu_1u by the proper label sequence 4024, relabel y by 3, and relabel the leaves adjacent to u by 0 and 1. As a result, $f(y_3) \neq 2$. Next, assign proper label sequence 3140 or 4204 to $y_3y_2y_1w'$ according to $f(y_3) = 3$ or 4. Finally, f can be extended to T after labeling the leaves adjacent to w'.

(7.5) w is incident to a closed 2-chain $[w'(2)w] = w'y_1y_2w$ such that w' is incident to an open 0-chain $(w_1(0)w']$ and a closed 1-chain $[w_2(1)w']$, where w_2 is a major handle. Note that both $[v(2)w]$ and $[w'(2)w]$ is of type (15, 2) and give 5 to w, respectively. Let $T' = T_{wx_2}(w) + [w(1)z_1] + [z_1(0)z_2) + [z_1(0)z_3)$, where $[w(1)z_1] = wz_0z_1$. Then $[z_1(1)w]$ gives 15 to v. By Lemma 5 and the induction hypothesis, T' is a good tree and it has a 4-$L(2,1)$-labeling f with $f(w) = 0$. Hence $f(z_1) = 4$ and $f(z_0) = 2$. Now we may label x_2 by 2 and assign proper label sequence 24024 to $x_2x_1vu_1u$ in T. Thus, f can be extended to T.

(C8) There is a 3-vertex v incident to three chains: a closed 1-chain $[u(1)v] = uu_1v$ such that u is a major handle; a closed 0-chain $[u'(0)v]$ such that u' is a major handle; a closed 2-chain $[v(2)w] = vx_1x_2w$.

(8.1) w is incident to an open 0-chain $(w'(0)w]$. Let P be the chain incident to v besides $[v(2)w]$ and $(w'(0)w]$. If P is open, then we can label all vertices of T easily. Assume $P = [w(k)w''] = wy_1 \cdots y_kw''$ is closed. Note that, $[u(1)v]$ and $[u'(0)v]$ give 15 and 6 to v respectively. Hence, $[v(2)w]$ is of type (3, 2) and it gives 10 to w. Furthermore, $[w(k)w'']$ is of type (10, k). We consider the following cases with different values of k:

(8.1-1) $k = 0$. Let $T' = T_{x_2x_1}(x_2)$. Then T' is a good tree and hence has a 4-$L(2,1)$-labeling f with $f(w) = 0$ by induction hypothesis. Therefore, we have $f(w'') = 4$. Relabel x_2 with 3 and w' with 2. Assign proper label sequence 31420 to $x_2x_1vu_1u$ in T and let $f(u') = 0$. Thus, f can be extended to T.

(8.1-2) $k = 1$. Let $T' = T_{x_2x_1}(x_2)$. Then T' is a good tree and hence has 4-$L(2,1)$-labeling f with $f(w) = 0$ by induction hypothesis. Thus, $f(w'') = 4$ and $f(y_1) = 2$. Relabel x_2 with 3 and w' with 4 and so, similar to (8.1-1), f can be extended to T.

(8.1-3) $k = 2$. Note that $[w(2)w'']$ is of type (10, 2) and gives 3 to w. Let $T' = T_{w''y_2}(w'') + [w''(4)z_1] + [z_1(0)z_2) + [z_1(1)z_3] + [z_3(0)z_4) + [z_3(0)z_5)$ and $z_0 \in [w''(4)z_1]$ be the neighbor of w''. Similar to (5.2-5), we have $f(z_0) \in \{2,4\}$ and hence we may let $f(y_2) = f(z_0)$. Then we assign proper label sequence 240 or 420 to y_2y_1w in T. Similar to (8.1-1), we can label x_2 with 3. As a result, f can be extended to T.

(8.1-4) $k = 3$. Then $[w(3)w'']$ is of type (10, 3) and gives 2 to w.
Let $T' = T_{y_3w''}(w'') + [w''(3)z_1] + [z_1(0)z_2) + [z_1(0)z_3)$ and $z_0 \in [w''(3)z_1]$ be the neighbor of w''. Same as (3.2) we have T' is a good tree and has a 4-$L(2,1)$-

labeling f with $f(w'') = 0$ and $f(z_0) = 3$ or 4. We assign proper label sequence 3140 or 4204 to $y_3y_2y_1w$ in T. Consequently, we can label u' with 2 and x_2 with 1 or 3 and so, similar to (8.1-1), f can be extended to T.

(8.1-5) $k = 4$, 5 or 6.

Let $T' = T_{y_4y_3}(y_4)$. Then T' is a good tree and hence has a 4-$L(2,1)$-labeling f with $f(y_5) \in \{0, 1, 2\}$ when $k = 5, 6$ and $f(w'') = 0$ when $k = 4$ by induction hypothesis.

If $f(y_5) = 0$ (or $f(w'') = 0$ when $k = 4$), then $f(y_4) = 2$, 3 or 4. We label the path $y_4y_3y_2y_1w$ in T with label sequence 24024, 31420 or 41304 accordingly.

If $f(y_5) = 1$, then $f(y_4) = 3$ or 4. We label the path $y_4y_3y_2y_1w$ in T with label sequence 30420 or 40240 accordingly.

If $f(y_5) = 2$, then $f(y_4) = 0$ or 4. We label the path $y_4y_3y_2y_1w$ in T with label sequence 03140 or 41304 accordingly.

For each case, $f(w) \in \{0, 4\}$ and $f(y_1) \notin \{1, 3\}$. Therefore, similar to (8.1-1), f can be extended to T.

(8.2) w is incident to a closed 0-chain $[w'(0)w]$, where w' is a major handle. Recall that $[v(2)w]$ gives ⑩ to w in T. Let $T' = T_{wx_2}(w) + [w(3)z_1] + [z_1(0)z_2) + [z_1(0)z_3)$ and $z_0 \in [w(3)z_1]$ be the neighbor of w. Clearly $[z_1(3)w]$ gives ② to w. By Lemma 5 and the induction hypothesis, T' is a good tree and has a 4-$L(2,1)$-labeling f with $f(w) = 0$ and hence $f(w') = 4$. Moreover, $f(z_0) \in \{1, 3\}$ and we label x_2 with $f(z_0)$ in T. Thus, similar to (8.1-1), f can be extended to T.

(8.3) w is incident to a closed 1-chain $[w'(1)w] = w'y_1w$, where w' is a major handle. Let $T' = T_{wx_2}(w) + [w(2)z_1] + [z_1(0)z_2) + [z_1(1)z_3] + [z_3(0)z_4) + [z_3(0)z_5)$ and $z_0 \in [w(3)z_1]$ be the neighbor of w. Recall that $[v(2)w]$ gives ⑩ to w in T. Clearly, $[z_1(2)w]$ gives ⑤ to w. By Lemma 5 and the induction hypothesis, T' is a good tree and has a 4-$L(2,1)$-labeling f with $f(w) = 0$ and hence $f(w') = 4$, $f(y_1) = 2$ and $f(z_0) \in \{1, 3\}$. Similar to (8.2), f can be extended to T.

(8.4) w is incident to a closed 3-chain $[w'(3)w] = w'y_1y_2y_3w$, where w' is a major handle. Let $T' = T_{wx_2}(w) + [w(3)z_1] + [z_1(0)z_2) + [z_1(0)z_3)$ and $z_0 \in [w(3)z_1]$ be the neighbor of w. Note that it is same as the tree T' in (C8.2). Hence, T' is a good tree and has a 4-$L(2,1)$-labeling f with $f(w) = 0$. Since $f(w), f(w'), f(z_1) \in \{0, 4\}$, $f(z_0), f(y_3) \in \{3, 4\}$. By Remark 3.1 we may exchange the labels of y_3 and z_0 if $f(y_3) = 3$. As a result, $f(y_3) = 4$. Reassign proper sequence 04204 to $wy_3y_2y_1w'$ and relabel the leaves of w'. Then x_2 may be assigned with 3 in T and, similar to (8.1-1), f can be extended to T.

(8.5) w is incident to a closed 2-chain $[w'(2)w] = w'y_1y_2w$ and w' is incident to an open 0-chain $(w_1(0)w']$ and a closed 1-chain $[w_2(1)w'] = w_2w''w'$, where w_2 is a major handle. Let $T' = T_{wx_2}(w) + [w(2)z_1] + [z_1(0)z_2) + [z_1(1)z_3] + [z_3(0)z_4) + [z_3(0)z_5)$ and $z_0 \in [w(2)z_1]$ be the neighbor of w. Since $[z_1(3)w]$ gives ⑤ to w, by Lemma 5 and the induction hypothesis, T' is a good tree and has a 4-$L(2,1)$-labeling f with $f(w) = 0$. Since $f(w) = 0$ and $f(w'), f(w_2), f(z_1), f(z_3) \in \{0, 4\}$, $f(z_0), f(y_2) \in \{2, 3\}$. By symmetry we may exchange the labels of y_2 and z_0 such that $f(y_2) = 2$ and

$f(z_0) = 3$ if necessary. Assign proper sequence 024024 to $wy_2y_1w'w''w_2$, and relabel the leaves of w' and w_2. Label x_2 with 3 in T and so f can be extended to T same as (8.1-1).

(8.6) w is incident to a closed 2-chain $[w'(2)w]$ and w' is incident to a closed 0-chain $[w_1(0)w']$ and a closed 1-chain $[w_2(1)w']$, where w_1 and w_2 are major handles. Note that $[v(2)w]$ gives ⑤ to w. On the other hand $[w_2(1)w']$ and $[w_1(0)w']$ give ⑮ and ⑥ to v, respectively. Then $[w'(2)w]$ is of type (③, 2) and gives ⑩ to w. Therefore, w is a bad vertex which is not a case.

(C9) There is a 3-vertex v incident to three chains: a closed 1-chain $[u(1)v]$, where u is a major handle; an open 0-chain $(y(0)v]$; and a closed 4-chain $[v(4)w] = vx_1x_2x_3x_4w$. Note that $[v(4)w]$ is of type (⑮, 4) and gives ③ to w.

(9.1) w is incident to an open 0-chain $(w'(0)w]$. Let $T' = T_{x_4x_3}(x_4)$. Then T' is a good tree and hence has a 4-$L(2,1)$-labeling f with $f(w) = 0$ by induction hypothesis. Exchange the labels of x_4 and w' if $f(x_4) = 3$. As a result, $f(x_4) \in \{2, 4\}$. Assign proper label sequence 2413024 or 4130420 to $x_4x_3x_2x_1vu_1u$. Thus, f can be extended to T.

(9.2) w is incident to a closed 0-chain $[w'(0)w]$, where w' is a major handle. Recall that $[v(4)w]$ gives ③ to w. Let $T' = T_{wx_4}(w) + [w(1)z_1] + [z_1(0)z_2) + [z_1(0)z_3)$ and $z_0 \in [w(1)z_1]$ be the neighbor of w. Note that $[z_1(1)w]$ gives ⑮ to w in T'. Hence T' is a good tree by Lemma 5 and has a 4-$L(2,1)$-labeling f with $f(w) = 0$ by induction hypothesis. Therefore, $f(z_1) = 4$ and $f(z_0) = 2$ and we label x_4 by $f(z_0) = 2$. Same as (9.1), f can be extended to T.

(9.3) w is incident to a closed 1-chain $[w'(1)w]$, where w' is a major handle. Let $T' = T_{wx_4}(w) + [w(0)z_1] + [z_1(0)z_2) + [z_1(0)z_3)$. Note that $[z_1(0)w]$ gives ⑥ to w in T'. So T' is a good tree by Lemma 5 and has a 4-$L(2,1)$-labeling f with $f(w) = 0$ by induction hypothesis. Then $f(z_1) = 4$. Hence, we may label x_4 by $f(z_1) = 4$. As the same as (9.1), f can be extended to T.

(9.4) w is incident to a closed 3-chain $[w'(3)w] = w'y_1y_2y_3w$, where w' is a major handle. Let $T' = T_{x_4x_3}(x_4)$. Then T' is a good tree and hence has a 4-$L(2,1)$-labeling f with $f(w) = 0$ by induction hypothesis. Whatever the labels of x_4 and y_3 are, it is easy to find a relabeling strategy such that $f(x_4) \in \{2, 4\}$ and $f(y_3) \in \{3, 4\}$. Same as (9.1), f can be extended to T.

(9.5) w is incident to a closed 2-chain $[w'(2)w] = w'y_1y_2w$ and w' is incident to an open 0-chain $(w_1(0)w']$ and a closed 1-chain $[w_2(1)w']$, where w_2 is a major handle. Let $T' = T_{x_4x_3}(x_4)$. Then T' is a good tree and has a 4-$L(2,1)$-labeling f with $f(w) = 0$ by induction hypothesis. Whatever the labels of x_4 and y_2 are, it is easy to find a relabeling strategy such that $f(x_4) \in \{2, 4\}$ and $f(y_2) \in \{2, 3\}$. Same as (9.1), f can be extended to T.

(9.6) w is incident to a closed 2-chain $[w'(2)w] = w'y_1y_2w$ and w' is incident to a closed 0-chain $[w_1(0)w']$ and a closed 1-chain $[w_2(1)w'] = w_2zw'$, where w_1 and w_2 are major

handles. Let $T' = T_{x_4x_3}(x_4)$. Then T' is a good tree and has a 4-$L(2,1)$-labeling f with $f(w) = 0$ by induction hypothesis. Since $\{f(w'), f(w_1)\} = \{0, 4\}$ and $f(z) = 2$, $f(y_1) \in \{1, 3\}$. Hence $f(y_2) \neq 2$. Moreover, $f(w') \in \{0, 4\}$ and $f(y_1) \neq 2$, $f(y_2) \neq 4$. As a result, $f(y_2) = 3$ and so $f(x_4) \in \{2, 4\}$. Same as (9.1), f can be extended to T.

(9.7) w is incident to a closed 4-chain $[w'(4)w]$ and w' is incident to an open 0-chain $(w_1(0)w']$ and a closed 1-chain $[w_2(1)w']$, where w_2 is a major handle. Let $T' = T_{x_4w}(w) + [w(0)z_1] + [z_1(0)z_2) + [z_1(0)z_3)$. Note that $[z_1(0)w]$ gives ⑥ to w in T'. So T' is a good tree by Lemma 5 and has a 4-$L(2,1)$-labeling f with $f(w) = 0$ by induction hypothesis. Thus, $f(z_1) = 4$ and we label x_4 by $f(z_1) = 4$. Same as (9.1), f can be extended to T.

(C10) There is a 3-vertex v incident to three chains: a closed 1-chain $[u(1)v] = uu_1v$, where u is a major handle; a closed 0-chain $[u'(0)v]$, where u' is a major handle; and a closed k-chain $[v(k)w] = vx_1x_2\cdots x_kw$, where $k \in \{3, 4, 5, 6\}$. Note that, $[u(1)v]$ and $[u'(0)v]$ give ⑮ and ⑥ to v, respectively. So $[v(k)w]$ is of type (③, k). We consider the following cases with different values of k:

(10.1) $k = 3$. Then $[v(3)w]$ gives ⑥ to w. Let $T' = T_{wx_3}(w) + [w(0)z_1] + [z_1(0)z_2) + [z_1(0)z_3)$. Same as (3.3) we have $f(w) = 0$ and $f(z_1) = 4$, where f is a 4-$L(2,1)$-labeling of T'. We assign proper label sequence 413024 to $x_3x_2x_1vu_1u$ and let $f(u') = 4$ in T. Thus, f can be extended to T.

(10.2) $k = 4$. Then $[v(4)w]$ gives ⑮ to w. Let $T' = T_{wx_4}(w) + [w(1)z_1] + [z_1(0)z_2) + [z_1(0)z_3)$ and $z_0 \in [w(1)z_1]$ be the neighbor of w. Same as (7.3-2), T' has a 4-$L(2,1)$-labeling f with $f(w) = 0$ and hence $f(z_0) = 2$. We assign proper label sequence 2413024 to $x_4x_3x_2x_1vu_1u$ and let $f(u') = 4$ in T. Thus, f can be extended to T.

(10.3) $k = 5$. Then $[v(5)w]$ gives ③ to w. Let $T' = T_{wx_5}(w) + [w(4)z_1] + [z_1(0)z_2) + [z_1(1)z_3] + [z_3(0)z_4) + [z_3(0)z_5)$ Same as (5.2-5) we obtain that T' has a 4-$L(2,1)$-labeling f with $f(w) = 0$ and $f(z_0) = 2$ or 4. Hence, assign proper label sequence 24031420 or 42031420 to $x_5\cdots x_1vu_1u$ and let $f(u') = 4$ in T. Thus, f can be extended to T.

(10.4) $k = 6$. Then $[v(6)w]$ gives ② to w. Let $T' = T_{x_6w}(w) + [w(3)z_1] + [z_1(0)z_2) + [z_1(0)z_3)$ and $z_0 \in [w(3)z_1]$ be the neighbor of w. Same as (3.2), T' has a 4-$L(2,1)$-labeling f with $f(w) = 0$ and $f(z_0) = 3$ or 4. Hence, assign proper label sequence 314031420 or 420413024 to $x_6\cdots x_1vu_1u$ and let $f(u') = 0$ or 4 in T accordingly. Thus, f can be extended to T. □

3.2 Necessity

Before proving the necessity, we explain the meaning of the weights given to the vertices.

Let K be a subtree of a tree T with $\Delta = 3$ containing a major vertex u. For any $w \in V(K)$, let

$$S_K^u(w) = \{f(w) \mid f \text{ is a 4-}L(2,1)\text{-labeling of } K \text{ such that } f(u) = 0\}.$$

Clearly $S_K^u(w) \subseteq \{2, 3, 4\}$ if w is a neighbor of u. If we required $f(u) = 4$, then we define

$$\overline{S}_K^u(w) = \{f(w) \mid f \text{ is a 4-}L(2,1)\text{-labeling of } K \text{ such that } f(u) = 4\}.$$

By symmetry, $\overline{S}^u_K(w) = \{4 - a \mid a \in S^u_K(w)\}$.

Let T be a tree with $\Delta = 3$ and $v \in V_3(T)$. For any uv-chain, let w_1 be the neighbor of v in the uv-chain. Suppose T has a 4-$L(2,1)$ labeling. By symmetry we assume that the label of v is 0. Let $K = T_v(vw_1)$. We define the following rules:

(1) *uv-chain gives ① to v means that* $S^v_K(w_1) = \{2, 3, 4\}$*;*

(2) *uv-chain gives ② to v means that* $S^v_K(w_1) = \{3, 4\}$*;*

(3) *uv-chain gives ③ to v means that* $S^v_K(w_1) = \{2, 4\}$*;*

(4) *uv-chain gives ⑤ to v means that* $S^v_K(w_1) = \{2, 3\}$*;*

(5) *uv-chain gives ⑥ to v means that* $S^v_K(w_1) = \{4\}$*;*

(6) *uv-chain gives ⑩ to v means that* $S^v_K(w_1) = \{3\}$*;*

(7) *uv-chain gives ⑮ to v means that* $S^v_K(w_1) = \{2\}$*.*

Remark 3.2 For Rule (1), since $S^v_K(w_1) \subseteq \{2, 3, 4\}$, it suffices to show that for each $a \in \{2, 3, 4\}$, there is a 4-$L(2,1)$-labeling f such that $f(w_1) = a$ and $f(v) = 0$. For Rules (5), (6) and (7), since we assume that T has a 4-$L(2,1)$-labeling, $S^v_K(w_1) \neq \varnothing$. Therefore, it suffices to show that $g(w_1) = 4$, 3, and 2, respectively for any 4-$L(2,1)$-labeling g of T when $g(v) = 0$.

From the definition, a weight ⓐ given to a major vertex v from the closed chain $[uv]$ depends on the length of $[uv]$ and the weights ⓑ and ⓒ given to u from the other two chains incident to u. It seems a 'binary operation' (ⓑ $*$ ⓒ $=$ ⓐ) on a special system. Now we define the meaning of such weights. Hence we need to show that this meaning is still closed (well-defined) according to the 'binary operation'.

Proof.

If u is a leaf in $T_v(vw_1)$, then $(uv]$ gives ① to v by Table 1. For each $a \in \{2, 3, 4\}$, it is easy to see that there exists a 4-$L(2,1)$ labeling f for $T_v(vw_1)$ such that $f(w_1) = a$. This satisfies Rule (1).

Now we assume u is a major vertex and let $[u(k)v] = uw_kw_{k-1}\cdots w_1v$ for $k \geq 0$.

When $k = 0$. Then $w_1 = u$ and $[u(0)v]$ gives ⑥ to v. For any 4-$L(2,1)$ labeling g, $g(v) = 0$ implies $g(w_1) = 4$. Thus it satisfies Rule (5).

When $k = 1$. $[u(1)v]$ gives ⑮ to v. For any 4-$L(2,1)$ labeling g with $g(v) = 0$, we deduce $g(w_1) = 2$ and $g(u) = 4$ that satisfies Rule (7).

Now we assume $k \geq 2$. Let P_1 and P_2 be the chains incident to u besides $[uv]$, $x_1 \in P_1$ and $x_2 \in P_2$ be the neighbors of u besides w_k, and P_1 and P_2 giving weight (y_1) and (y_2) to u, respectively.

Properties on labels of w_k:

Now we consider the subtree $H = T_{w_k}(w_ku)$.

(P1) If $[u(k)v]$ is of type (①$, k$), then $y_1, y_2 \in \{1, 2, 3, 5\}$ and $\gcd(y_1, y_2) = 1$. We want to show that $S^u_H(w_k) = \{2, 3, 4\}$.

From Rules (1)-(4) we have $S^u_{T_1}(x_1) \cup S^u_{T_2}(x_2) = \{2,3,4\}$, where $T_1 = T_u(ux_1)$ and $T_2 = T_u(ux_2)$. Note that, if $y_1 = y_2 = 1$, then by Rule (1), $S^u_{T_1}(x_1) = S^u_{T_2}(x_2) = \{2,3,4\}$. Otherwise $S^u_{T_1}(x_1) \neq S^u_{T_2}(x_2)$.

Let $a \in \{2,3,4\}$. Without loss of generality, we may assume $a \in S^u_{T_1}(x_1)$. It is easy to see that $|S^u_{T_1}(x_1)| \geq 2$ and $|S^u_{T_2}(x_2)| \geq 2$. So we may choose $b \in S^u_{T_1}(x_1) \setminus \{a\}$. Since either $S^u_{T_1}(x_1) = S^u_{T_2}(x_2) = \{2,3,4\}$ or $S^u_{T_1}(x_1) \neq S^u_{T_2}(x_2)$, there exists $c \in S^u_{T_2}(x_2) \setminus \{a,b\}$. According to our rules, there are 4-$L(2,1)$-labelings g_1 of T_1 and g_2 of T_2 such that $g_1(x_1) = b$ and $g_2(x_2) = c$. Note that $g_1(u) = g_2(u) = 0$. Label w_k by a in H together with g_1 and g_2 deduce that $S^u_H(w_k) = \{2,3,4\}$.

(P2) If $[u(k)v]$ is of type $(②, k)$, then y_1 and y_2 share the unique common prime divisor 2. By Rules (2), (5) and (6), we have $g_1(x_1), g_2(x_2) \neq 2$ for any 4-$L(2,1)$-labelings g_1 of $T_u(ux_1)$ and g_2 of $T_u(ux_2)$ with $g_1(u) = g_2(u) = 0$. Thus, w_k can only be labeled by 2 in H and so $S^u_H(w_k) = \{2\}$. Similarly, we can show that $S^u_H(w_k) = \{3\}$ and $S^u_H(w_k) = \{4\}$ if $[u(k)v]$ is of type $(③, k)$ and $(⑤, k)$, respectively.

(P3) If $[u(k)v]$ is of type $(⑥, k)$, without loss of generality, assume $y_1 = 6$ and $y_2 \in \{1,5\}$. By Rule (5), $g_1(x_1) = 4$ for any 4-$L(2,1)$-labeling g_1 of $T_u(ux_1)$. Hence $S^u_H(w_k) \subseteq \{2,3\}$.

For any $a \in \{2,3\}$. By Rules (1) and (4), $\{2,3\} \subseteq S^u_{T_u(ux_2)}(x_2)$. There is a 4-$L(2,1)$-labeling g_2 of $T_u(ux_2)$ such that $g_2(x_2) = b$ and $g_2(u) = 0$, where $b \in S^u_{T_u(ux_2)}(x_2) \setminus \{a\}$. Now we label w_k by a in H and so $\{2,3\} \subseteq S^u_H(w_k)$. Therefore, $S^u_H(w_k) = \{2,3\}$. Similarly, we can show that $S^u_H(w_k) = \{2,4\}$ and $S^u_H(w_k) = \{3,4\}$ if $[u(k)v]$ is of type $(⑩, k)$ and $(⑮, k)$, respectively.

Finally we only need to verify every type of chain in Table 1 by using the above properties.

(1) v receives ①. We only need to show that for each $a \in \{2,3,4\}$, there is a 4-$L(2,1)$-labeling for $T_v(vw_1)$ such that $f(w_1) = a$ and $f(v) = 0$.

• $[u(k)v]$ is of type $(①, 2)$. According to the prescribed labels of w_1, we label vw_1w_2u by 0240, 0314 and 0420, respectively. By (P1), since $S^u_H(w_2) = \{2,3,4\}$, where $H = T_{w_2}(w_2u)$, we have a required labeling for the subtree $T_v(vw_1)$.

• $[u(k)v]$ is of type $(①, 4^+)$. We first assume that $k \geq 5$. No matter what is the label of w_1 has been assigned, there are at most twelve possible cases for the labels of $w_{k-4}w_{k-3}$. Namely, 02, 03, 04, 13, 14, 20, 24, 30, 31, 40, 41 and 42. The following six labelings and their symmetric labelings for $w_{k-4}w_{k-3}w_{k-2}w_{k-1}w_ku$ cover all those case: 024130, 031420, 041304, 130420, 140314, 204130 and the result follows. When $k = 4$, only the first three cases are needed to consider and we obtain the result similar to the case above.

• Other types described at the first row of Table 1, which are not mentioned here, can be verified by similar method using (P2) or (P3).

(2) v receives ②. We need to show that $S^v_{T_v(vw_1)}(w_1) = \{3,4\}$.

• $[u(k)v]$ is of type $(①, 3)$. We label $vw_1w_2w_3u$ by 03140 and 04130 accordingly. By (P1), we have a required labeling. On the other hand, if there is a 4-$L(2,1)$-labeling g such that $g(w_1) = 2$ and $g(v) = 0$, then $g(w_2) = 4$ and $g(u) = 0$. As a result, $g(w_3)$ cannot be defined.

• $[u(k)v]$ is of type $(⑩, 3)$. We label $vw_1w_2w_3u$ by 03140 and 04204 accordingly. By (P3), we have a required labeling. On the other hand, if there is a 4-$L(2,1)$-labeling g such that $g(w_1) = 2$ and $g(v) = 0$, then it is same as the case above.

• $[u(k)v]$ is of type $(②, 5)$. We label $vw_1w_2w_3w_4w_5u$ by 0314024 and 0420420. By (P2), we have a required labeling. On the other hand, if there is a 4-$L(2,1)$-labeling g such that $g(w_1) = 2$ and $g(v) = 0$, then $g(w_2) = 4$, $(g(w_3), g(w_4)) = (0,2)$, $(0,3)$ or $(1,3)$, which contradicts (P2).

• Other types described at the second row of Table 1, which are not mentioned here, can be verified by similar method using (P2) or (P3).

(3) v receives ③. We need to show that $S^v_{T_v(vw_1)}(w_1) = \{2, 4\}$.

• $[u(k)v]$ is of type $(②, 6)$. We label $vw_1w_2w_3w_4w_5w_6u$ by 02413024 and 04130420 respectively. By (P2), we have a required labeling. On the other hand, if there is a 4-$L(2,1)$-labeling g such that $g(w_1) = 3$ and $g(v) = 0$, then $g(w_2) = 1$, $g(w_3) = 4$ and $g(w_4) \in \{0, 2\}$. But (P2) implies $g(w_6) = 2$ and so $g(w_5)$ cannot be defined.

• $[u(k)v]$ is of type $(⑩, 2)$. We label vw_1w_2u by 0240 and 0420 respectively. By (P3), we have a required labeling. On the other hand, if there is a 4-$L(2,1)$-labeling g such that $g(w_1) = 3$ and $g(v) = 0$, then $g(w_2) = 1$ and $g(u) = 4$. However, (P3) implies $g(w_2) \in \{2, 0\}$ and contradiction occurs.

• $[u(k)v]$ is of type $(⑮, 4)$. We label $vw_1w_2w_3w_4u$ by 024130 and 041304 respectively. By (P3), we have a required labeling. On the other hand, if there is a 4-$L(2,1)$-labeling g such that $g(w_1) = 3$ and $g(v) = 0$, then $g(w_2) = 1$, $g(w_3) = 4$ and $g(w_4) \in \{0, 2\}$. Since $g(w_3) = 4$, $g(u) = 0$ and hence $g(w_4) = 2$, which contradicts (P3).

• $[u(k)v]$ is of type $(③, 5)$. We label $vw_1w_2w_3w_4w_5u$ by 0240314 and 0420314 respectively. By (P2), we have a required labeling. On the other hand, if there is a 4-$L(2,1)$-labeling g such that $g(w_1) = 3$ and $g(v) = 0$, then $g(w_2) = 1$, $g(w_3) = 4$ and $g(w_4) \in \{0, 2\}$. By (P2), $(g(w_5), g(u)) = (3, 0)$ or $(1, 4)$. This is a contradiction.

(4) v receives ⑤. We need to show that $S^v_{T_v(vw_1)}(w_1) = \{2, 3\}$.

• $[u(k)v]$ is of type $(⑤, 5)$. We label $vw_1w_2w_3w_4w_5u$ by 0241304 and 0314204 respectively. By (P2), we have a required labeling. On the other hand, if there is a 4-$L(2,1)$-labeling g such that $g(w_1) = 4$ and $g(v) = 0$, then $(g(w_2), g(w_3), g(w_4)) = (1, 3, 0)$, $(2, 0, 3)$ or $(2, 0, 4)$. According to (P2), $(g(w_5), g(u)) = (4, 0)$ or $(0, 4)$ and so g does not exist.

• $[u(k)v]$ is of type $(②, 4)$. We label $vw_1w_2w_3w_4u$ by 024024 and 031420 respectively. By (P2), we have a required labeling. On the other hand, if there is a 4-$L(2,1)$-labeling g such that $g(w_1) = 4$ and $g(v) = 0$, then $(g(w_2), g(w_3), g(w_4)) = (1, 3, 0)$, $(2, 0, 3)$ or $(2, 0, 4)$. As a result, $g(u)$ cannot be defined.

- $[u(k)v]$ is of type (⑥, 4). We label $vw_1w_2w_3w_4u$ by 024130 and 031420 respectively. By (P3), we have a required labeling. On the other hand, if there is a 4-$L(2,1)$-labeling g such that $g(w_1) = 4$ and $g(v) = 0$, then it is the same the case above.

- $[u(k)v]$ is of type (⑮, 2). We label vw_1w_2u by 0240 and 0314 respectively. By (P3), we have a required labeling. On the other hand, if there is a 4-$L(2,1)$-labeling g such that $g(w_1) = 4$ and $g(v) = 0$, then $g(u) = 4$ and $g(w_2) = 2$, which contradicts (P3).

(5) v receives ⑥. We only need to show that $g(w_1) = 4$ for any 4-$L(2,1)$-labeling g of T with $g(v) = 0$.

- $[u(k)v]$ is of type (②, 2). By (P2), $g(w_2) = 2$ and hence $g(w_1) = 4$.

- $[u(k)v]$ is of type (③, 3). By (P2), the label sequence of uw_3w_2 under g is 031 or 413. Therefore, $g(w_1) = 4$.

- $[u(k)v]$ is of type (⑤, 4). By (P2), the label sequence of $uw_4w_3w_2$ is 0413, 0420, 4031 or 4024. It is easy to check that only the third case exists when $g(v) = 0$. It implies that $g(w_1) = 4$.

- $[u(k)v]$ is of type (⑥, 3). By (P3), the label sequence of uw_3w_2 is 031, 024, 420 or 413. It is easy to check that only the first case exists when $g(v) = 0$. Hence $g(w_1) = 4$.

(6) v receives ⑩. We only need to show that $g(w_1) = 3$, for any 4-$L(2,1)$-labeling g of T with $g(v) = 0$.

- $[u(k)v]$ is of type (③, 2). By (P2), $(g(u), g(w_2)) = (0, 3)$ or $(4, 1)$. Since $g(v) = 0$, only the last case exists and hence $g(w_1) = 3$.

(7) v receives ⑮. We only need to show that $g(w_1) = 2$, for any 4-$L(2,1)$-labeling g of T with $g(v) = 0$.

- $[u(k)v]$ is of type (③, 4). By (P2), the label sequence of $uw_4w_3w_2$ is 0314 or 4130. Since $g(v) = 0$, only the first case exists and hence $g(w_1) = 2$.

- $[u(k)v]$ is of type (⑤, 2). By (P2), the label sequence of uw_2 is 04 or 40. Since $g(v) = 0$, only the first case exists and hence $g(w_1) = 2$. □

The proof of necessity.

Let f be a 4-$L(2,1)$-labeling of T using label set $\mathcal{B} = \{0, 1, 2, 3, 4\}$. Assume the contrary that T is a bad tree, that is, T contains a bad subtree T^* such that T^* satisfies the conditions in Definition 2. Let u be the bad vertex of T^*. Note that u is a major vertex, hence we assume that $f(u) = 0$. Consider the following cases depending on the reason of u being the bad vertex.

(1) There is a closed chain $[u(3)v] = ux_1x_2x_3v$ such that $[u(3)v]$ is of type (②, 3).

Let P_1 and P_2 be the chains incident to u besides $[uv]$ and let $w_1 \in P_1$ and $w_2 \in P_2$ be the neighbors of u. Since $[u(3)v]$ is of type (②, 3), the greatest common divisor of the weights from P_1 and P_2 to u is 2. Thus, $f(w_1), f(w_2) \neq 2$ by Rules (2), (5) and (6). This implies that $f(x_1) = 2$ and $f(x_2) = 4$ and hence $f(v) = 0$. Thus f is not a 4-$L(2,1)$-labeling of T because $f(x_3)$ cannot be defined, contradiction occurs.

(2) There are two chains incident to u giving the same weight ⑥, ⑩ or ⑮ to u.

Let P_1 and P_2 be these chains and $w_1 \in P_1$ and $w_2 \in P_2$ be the neighbors of u.

If P_1 and P_2 give the same weight ⑥ to u, then $f(w_1), f(w_2) \in \{0, 4\}$ by Rule (5). However, $f(u) = 0$ which yields a contradiction.

If P_1 and P_2 give the same weight ⑩ to u, then $f(w_1), f(w_2) \in \{1, 3\}$ by Rule (6). However, $f(u) = 0$ which yields a contradiction.

If P_1 and P_2 give the same weight ⑮ to u, then $f(w_1) = f(w_2) = 2$ by Rule (7). This is a contradiction.

(3) The weights from all three chains incident to u have the greatest common divisor 2, 3 or 5.

Let P_1, P_2 and P_3 be these chains and $w_1 \in P_1$, $w_2 \in P_2$ and $w_3 \in P_3$ be the neighbors of u. Let d be the greatest common divisor of these three weights that u receives.

According to Rules (2) to (7), if $d = 2$, then $f(w_1), f(w_2), f(w_3) \in \{0, 1, 3, 4\}$; if $d = 3$, then $f(w_1), f(w_2), f(w_3) \in \{0, 2, 4\}$; if $d = 5$, then $f(w_1), f(w_2), f(w_3) \in \{1, 2, 3\}$. But $f(u) = 0$, which yields a contradiction.

This completes the proof. □

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Appendix